Ranking the sequences of fuzzy values

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Abstract

When ranking fuzzy values, there is an ambiguity in selecting the most-preferred ranked sequence from a collection of candidate sequences because every sequence has a possibility to which the preference relations in the sequence are correct. In this paper, we propose a method for ranking the sequences of fuzzy values that assigns a preference degree to each ranked sequence. Each preference degree indicates the relative preference of the given sequence over the other sequences. The proposed method gives information regarding which sequence could be classified as the most-preferred sequence and which sequences as alternatives.

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1. Introduction

The comparison and ranking of fuzzy values is an important aspect of various fuzzy applications. However, the ambiguity inherent in fuzzy values leads to the ambiguity in the comparison and ranking, which makes it difficult to find crisp and definite preference relations.
To solve this problem, researchers developed various methods in which a preference degree is assigned to each fuzzy value. The preference degree denotes the degree to which the given fuzzy value is preferred over the other ones. These methods mainly fall into one of two categories [1]: either they use a ranking function that maps each fuzzy value into a crisp value, or they obtain a fuzzy set of possible alternatives for the largest among fuzzy values. By ordering the fuzzy values according to their preference degrees, a ranked sequence can be generated.

From a given set of fuzzy values, however, multiple number of ranked sequences can be generated. And each sequence has a possibility to which the preference relations in the sequence are correct. That is, even though we select the most-preferred sequence according to the preference degrees of the fuzzy values, other sequences could still be viable alternatives. However, previous researches give insufficient information to determine which sequences are potential alternatives.

To deal with this problem, we propose a method that assigns a preference degree to each ranked sequence instead of each fuzzy value. For calculating the preference degree, we define a measure called the preference degree of sequence (PDS) whose value indicates the relative preference of the given sequence over the other sequences. This measure is an extension of the preference degree of comparison (PDC) that was proposed by Lee et al. [2] to compare fuzzy values. In the present work, some properties of the PDS are analyzed, including its relation to the PDC.

This paper is arranged as follows: In Section 2, we give a brief review of previous works on the ranking of fuzzy values. The proposed method for ranking the sequences of fuzzy values is introduced in Section 3 and its properties are analyzed. Finally, we present our conclusions in Section 4.

2. Related works

2.1. Ranking fuzzy values

A fuzzy value is a fuzzy set defined in the real number domain \( \mathbb{R} \). Let us suppose we have \( n \) fuzzy values \( A_i, i \in N = 1, 2, \ldots, n \):

\[
A_i = \{(x, \mu_{A_i}(x)) | \mu_{A_i}(x) \in [0, 1]\},
\]

where \( \mu_{A_i}(x) \) is the membership function of \( A_i, x \in \mathbb{R} \).

Two approaches have been predominantly used to solve the problem of ranking fuzzy values [1]. The first approach is to define a ranking function \( F \) that maps each fuzzy value into a crisp value in the real number domain, where a natural order exists [3–5]; that is,
where $\mathcal{A}$ is the set of fuzzy values $A_1, \ldots, A_n$, and $F$ is such that

- $F(A_i) < F(A_j)$ implies $A_i < A_j$,
- $F(A_i) = F(A_j)$ implies $A_i = A_j$,
- $F(A_i) > F(A_j)$ implies $A_i > A_j$.

A simple example of the mapping function is

$$F(A_i) = \frac{\int g(x)\mu_{A_i}(x)\,dx}{\int \mu_{A_i}(x)\,dx},$$

where the weight $g(x)$ is a measure of the importance of the value $x$ [3].

The second approach is to obtain a fuzzy set $L$ of possible alternatives for the largest among fuzzy values such that

$$L = \{(A_i, \mu_L(A_i))\}, \quad i \in N,$$

where $\mu_L(A_i)$ is the preference degree to which the fuzzy value $A_i$ is considered to be the most preferred. An example of the preference degree is proposed as follows [6]:

$$\mu_L(A_i) = \sup_{x_i} \min_{x_j \neq i} \left[ \mu_{A_i}(x_i), \min_{x_j \geq x_i} \max_{x_i, x_j} \mu_{A_j}(x_j) \right].$$

The definition of the mapping function or preference degree may vary with respect to the criteria of evaluation. Therefore, approaches have been developed that provide multiple indices to represent the location of each fuzzy value in various aspects with multiple criteria [7–9].

### 2.2. The preference degree of comparison

As mentioned in Section 2.1, numerous approaches have been used to calculate the preference degrees of fuzzy values. In this section, we introduce a measure called the preference degree of comparison (PDC) that was proposed by Lee et al. [2] to calculate the preference degrees in comparing fuzzy values.

Let $A$ and $B$ be fuzzy values and $\Diamond$ be an arithmetic comparison relation such as $<, >, =, \leq$ or $\geq$. The satisfaction degree $s(A \Diamond B)$ denotes the degree to which the arithmetic comparison relation $\Diamond$ for $A$ and $B$ is satisfied, or, the degree to which the proposition $A \Diamond B$ is true. The value of the satisfaction degree lies in the range $[0, 1]$.

The three satisfaction degrees $s(A_i < A_j)$, $s(A_i = A_j)$, and $s(A_i < A_j)$ for discrete fuzzy values $A_i$ and $A_j$ are defined as
where $\otimes$ is a $t$-norm operator satisfying the following condition:

$$a \otimes b > 0 \quad \text{if} \quad a > 0 \quad \text{and} \quad b > 0.$$  

(4)

The PDC $p_c$ with the satisfaction degree $s$ is defined as follows:

$$p_c(A_i, A_j) = s(A_i > A_j) + \frac{s(A_i = A_j)}{2}.$$  

(5)

Because the definitions of the satisfaction degree and PDC are only applicable to fuzzy values defined on a discrete domain, methods have been developed to apply this approach to fuzzy values defined on a continuous domain [10] or to type-2 fuzzy values of various kinds [11,12].

3. Ranking the sequences of fuzzy values

The methods developed to date for ranking fuzzy values enable the determination of the most preferred value and the alternative values, but they give no information on which ranked sequence is the most preferred sequence or on alternative sequences. To address this shortcoming, we herein propose a method for ranking the sequences instead of the fuzzy values. The proposed method is developed by extending the PDC concept to the ranking of sequences of fuzzy values.

3.1. The preference degree of sequence

Let us consider a set of $n$ fuzzy values $A_i, i \in N = \{1, 2, \ldots, n\}$. We assume that all the given fuzzy values are defined on a discrete domain. If any of them is defined on a continuous domain, it can be discretized using a discretization method for a fuzzy set [2].

**Definition 1.** An occurrence is one of the possible combinations of the actual values of $A_1$ to $A_n$, that is,

$$o = (x_1, x_2, \ldots, x_n), \quad x_i \in A_i.$$

The $i$th value of an occurrence $o$ is denoted as $o(i)$. 
In the case of ranking \( n \) fuzzy values, the number of possible occurrences is
\[
n(O) = n(A_1) \times n(A_2) \times \cdots \times n(A_n),
\]
where \( O \) is the set of all possible occurrences and \( n(A) \) is the cardinality of set \( A \).

When there is a preference relation \( \diamond \) between \( o(i) \) and \( o(j) \), the set of all occurrences for which this relation is true is denoted as \( O(A_i \diamond A_j) \), \( O(A_i \diamond A_j) \subseteq O \), where \( \diamond \) is an arithmetic comparison relation such as \( <, >, =, \leq, \text{or} \geq \). For example,
\[
O(A_i \geq A_j) = \{ o_k \mid o_k(i) \geq o_k(j), \; o_k \in O \}.
\]
Other sets such as \( O(A_i > A_j) \), \( O(A_i = A_j) \), or \( O(A_i \leq A_j) \) can be defined in a similar way.

Each occurrence has a weight indicating its relative preference compared to the other occurrences; we refer to this weight as the occurrence degree. It is defined as the aggregation of the membership degrees using a \( t \)-norm operator.

**Definition 2.** The occurrence degree of an occurrence \( (x_1, x_2, \ldots, x_n) \) is defined as
\[
w(x_1, x_2, \ldots, x_n) = \mu_{A_1}(x_1) \otimes \mu_{A_2}(x_2) \otimes \cdots \otimes \mu_{A_n}(x_n),
\]
where \( \otimes \) is a \( t \)-norm operator satisfying the condition given in Eq. (4). The value of the occurrence degree lies in the range \([0, 1]\).

**Definition 3.** A ranked sequence is a sequence of fuzzy values ordered according to their preference relations.

For example, a ranked sequence \( r \) of the form
\[
r = (A_2, A_1, A_3)
\]
indicates that \( A_2 \) is preferred over \( A_1 \) and \( A_1 \) is preferred over \( A_3 \). We interpret the preference relation in a ranked sequence to be the relation in which the preferred fuzzy value is greater than or equal to the other fuzzy value; thus the interpretation of the ranked sequence \( (A_2, A_1, A_3) \) is \( A_2 \geq A_1 \geq A_3 \).

There are \( n! \) ranked sequences of \( n \) fuzzy values. We denote the set of all ranked sequences as \( R \) and each ranked sequence as \( r_i \), \( r_i \in R, \; i = 1, \ldots, n! \). Moreover, \( \gamma(A_i, A_j) \) denotes the set of ranked sequences in which \( A_i \) is preferred over \( A_j \) in the sequence, where \( \gamma(A_i, A_j) \subseteq R \).

Because an occurrence can be looked upon as an instance of a ranked sequence, we can build a set of occurrences satisfying the preference relation of a ranked sequence. We call this set the set of matching occurrences of a ranked sequence \( r \). It is denoted as \( \eta(r) \).
Example 1. Let us consider three fuzzy values $A_1$, $A_2$, and $A_3$ that are given as shown in Fig. 1. For the following ranked sequence $r$ and two occurrences $o_1$ and $o_2$, 

$$r = (A_3, A_1, A_2), \quad o_1 = (5, 2, 7), \quad o_2 = (3, 4, 6),$$

the following relations hold:

$$o_1 \in \eta(r), \quad o_2 \notin \eta(r)$$

because the occurrence $o_1$ satisfies the preference relation $A_3 \succeq A_1 \succeq A_2$ whereas $o_2$ does not.

An occurrence can satisfy more than one ranked sequence. Here, the number of ranked sequences that an occurrence $o$ satisfies is denoted as $m(o)$, where $m(o) \in \{1, 2, \ldots, n\}$.

Using this notation, the preference degree of sequence (PDS) is formulated as follows:
**Definition 4.** The PDS of a ranked sequence \( r \), \( p_s(r) \), is defined as

\[
p_s(r) = \frac{\sum_{\forall l, o_l \in r} (w(o_l) / m(o_l)) \cdot \sum_{\forall l} w(o_l)}{\sum_{\forall l} w(o_l)}.
\] (7)

**Definition 5.** The result of ranking the sequences of fuzzy values is given in the form of a fuzzy set defined as

\[
\{(r_i, p_s(r_i)) | r_i \in R\},
\]

where \( p_s(r_i) \) is the PDS of a sequence \( r_i \), \( p_s(r_i) \in [0, 1] \).

**Example 2.** In this example, we rank three fuzzy values \( A_1, A_2, \) and \( A_3 \), where

\[
A_1 = \{(2, 0.9), (4, 0.3)\},
\]
\[
A_2 = \{(3, 0.4), (4, 0.5)\},
\]
\[
A_3 = \{(4, 0.7), (5, 0.2)\}.
\]

Each fuzzy value has two actual values, which means that the total number of occurrences is 8. If we select the normal multiplication operator as the \( t \)-norm operator, the occurrence degree of each occurrence is found to be

\[
w(2, 3, 4) = 0.9 \times 0.4 \times 0.7 = 0.252,
\]
\[
w(2, 3, 5) = 0.072,
\]
\[
w(2, 4, 4) = 0.315,
\]
\[
w(2, 4, 5) = 0.090,
\]
\[
w(4, 3, 4) = 0.084,
\]
\[
w(4, 3, 5) = 0.024,
\]
\[
w(4, 4, 4) = 0.105,
\]
\[
w(4, 4, 5) = 0.030
\]

and

\[
\sum_i w(o_i) = 0.972.
\]

The number of ranked sequences is 6, and the set of matching occurrences of each ranked sequence is

\[
\eta(r_1) = \eta(A_1, A_2, A_3) = \{(4, 4, 4)\},
\]
\[
\eta(r_2) = \eta(A_1, A_3, A_2) = \{(4, 3, 4), (4, 4, 4)\},
\]
\[
\eta(r_3) = \eta(A_2, A_1, A_3) = \{(4, 4, 4)\},
\]
\[
\eta(r_4) = \eta(A_2, A_3, A_1) = \{(2, 4, 4), (4, 4, 4)\},
\]
\( \eta(r_5) = \eta(A_3, A_1, A_2) = \{(4, 3, 4), (4, 3, 5), (4, 4, 4), (4, 4, 5)\} \),
\( \eta(r_6) = \eta(A_3, A_2, A_1) = \{(2, 3, 4), (2, 3, 5), (2, 4, 4), (2, 4, 5), (4, 4, 4), (4, 4, 5)\} \).

The _preference degree_ of each sequence can be calculated as follows:

\[
p_k(r_1) = \left( \frac{1}{6} w(4, 4, 4) \right) / \sum_i w(o_i) = 0.018,
\]

\[
p_k(r_2) = \left( \frac{1}{2} w(4, 3, 4) + \frac{1}{6} w(4, 4, 4) \right) / \sum_i w(o_i) = 0.061,
\]

\[
p_k(r_3) = \left( \frac{1}{6} w(4, 4, 4) \right) / \sum_i w(o_i) = 0.018,
\]

\[
p_k(r_4) = \left( \frac{1}{2} w(2, 4, 4) + \frac{1}{6} w(4, 4, 4) \right) / \sum_i w(o_i) = 0.180,
\]

\[
p_k(r_5) = \left( \frac{1}{2} w(4, 3, 4) + w(4, 3, 5) + \frac{1}{6} w(4, 4, 4) + \frac{1}{2} w(4, 4, 5) \right) / \sum_i w(o_i) = 0.101,
\]

\[
p_k(r_6) = \left( w(2, 3, 4) + w(2, 3, 5) + \frac{1}{2} w(2, 4, 4) + w(2, 4, 5) + \frac{1}{6} w(4, 4, 4)
\]

\[+ \frac{1}{2} w(4, 4, 5) \right) / \sum_i w(o_i) = 0.621.\]

Thus, the result of ranking the sequences according to the PDS is

\{
((A_3, A_2, A_1), 0.621), ((A_2, A_3, A_1), 0.180),
((A_1, A_3, A_2), 0.061), ((A_3, A_1, A_2), 0.101),
((A_1, A_2, A_3), 0.018), ((A_2, A_1, A_3), 0.018)\}.

This result indicates that the ranked sequence \((A_3, A_2, A_1)\) has the greatest preference degree and \((A_2, A_3, A_1)\) is the next-best alternative.

### 3.2. Properties of the PDS

The PDS has the following properties. First, we consider two properties related to the boundaries of the PDS.

**Property 1.** \( \sum_i p_k(r_i) = 1, r_i \in R. \)
Proof. If we define the set \( M_i \subseteq O \) as
\[
M_i = \{ o \mid m(o) = i, \ o \in O \}
\]
then
\[
\sum_i w(o_i) = \sum_{o_i \in M_1} w(o_i) + \sum_{o_i \in M_2} w(o_i) + \cdots + \sum_{o_i \in M_n} w(o_i)
\]
\[
= \sum_{o_i \in M_1} w(o_i) + 2\left( \frac{1}{2} \sum_{o_i \in M_2} w(o_i) \right) + \cdots + n! \left( \frac{1}{n!} \sum_{o_i \in M_n} w(o_i) \right)
\]
\[
= \sum_{o_i \in \eta(r_1)} w(o_i) + \sum_{o_i \in \eta(r_2)} w(o_i) + \cdots + \sum_{o_i \in \eta(r_n)} w(o_i)
\]
\[
= \sum_i \sum_{j, o_j \in \eta(r_j)} \frac{w(o_j)}{m(o_j)}
\]
and therefore
\[
\sum_i p_s(r_i) = 1. \]

Property 2. \( 0 \leq p_s(r) \leq 1 \).

Proof. \( p_s(r) \geq 0 \) because \( w(o) \geq 0 \) and \( m(o) \geq 0 \) for every \( o \in O \). And \( p_s(r) \leq 1 \) according to \( s(r) \geq 0 \) and Property 1. Therefore \( 0 \leq p_s(r) \leq 1 \). \( \square \)

Now we consider a condition under which the PDS is 0. Let \( A_i \) and \( A_j \) be two fuzzy values. We define \( A_i \) to be strictly dominant over \( A_j \) if and only if
\[
\inf(\text{supp}(A_i)) > \sup(\text{supp}(A_j)),
\]
where \( \text{supp}(A) \) is a support of a fuzzy set \( A \).

Property 3. If \( r \in \gamma(A_j, A_i) \) and \( A_i \) is strictly dominant over \( A_j \), then \( p_s(r) = 0 \).

Proof. If a ranked sequence \( r \in \gamma(A_j, A_i) \) and \( A_i \) is strictly dominant over \( A_j \), there is no occurrence that can satisfy the preference relation of the ranked sequence. In other words, \( \{ o \mid o \in \eta(r) \} = \emptyset \) and therefore \( p_s(r) = 0 \). \( \square \)

In addition to the above properties, we found the following relation between the PDS and the PDC described in Section 2.1.

Property 4. The PDS and PDC satisfy the following relation:
Proof. The summation of PDSs is

$$p_c(A_i, A_j) = \sum_{\forall k, r_k \in \gamma(A_i, A_j)} p_s(r_k).$$

(8)

If we calculate the value of $m(o_l)$ of each set,

$$m(o_l) = 1 \quad \forall o_l \in O(A_i > A_j),$$

$$m(o_l) = 2 \quad \forall o_l \in O(A_i = A_j).$$

Therefore,

$$\left( \sum_{\forall k, r_k \in \gamma(A_i, A_j)} \sum_{o_l \in \eta(r_k)} w(o_l)m(o_l) \right) / \left( \sum_l w(o_l) \right)$$

$$= \left( \sum_{o_l \in O(A_i > A_j)} w(o_l) + \frac{1}{2} \sum_{o_l \in O(A_i = A_j)} w(o_l) \right) / \left( \sum_l w(o_l) \right)

= \frac{\sum_{o_l \in O(A_i > A_j)} w(o_l)}{\sum_l w(o_l)} + \frac{1}{2} \frac{\sum_{o_l \in O(A_i = A_j)} w(o_l)}{\sum_l w(o_l)} = s(A_i > A_j) + \frac{1}{2} s(A_i = A_j)

= p_c(A_i, A_j).$$

Based on these properties, the PDSs can be calculated indirectly using the PDCs. First, one linear equations of PDSs can be found according to Property 1. And if we calculate the PDCs between every pair of fuzzy values, we can obtain $C(n, 2)$ linear equations of PDSs using Property 4, where $C(n, m)$ is the combination. Therefore, if we can find $(n! - (C(n, 2) + 1))$ PDSs that have values of 0 according to Property 3, we can calculate the values of the PDSs indirectly by solving the given linear equations.
4. Conclusion

A method for ranking the sequences of fuzzy values is proposed. The proposed method determines the preference degree of each sequence, which indicates the relative preference of the given sequence over the other sequences. These preference degrees can be used to determine alternative most-preferred sequences from among the ranked sequences of a given set of fuzzy values. We also established the condition under which the preference degree of sequence can be calculated using the preference degree of comparison, an approach which can reduce the computational complexity.

The proposed method can be applied to the problems where the ranked sequence is required to satisfy some conditions. After evaluating each ranked sequence, we can choose a sequence as the best solution that has the greatest preference degree while satisfying the given conditions.

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